Answer **four** problems. (If you turn in more, the first four will be graded.) Put your answers in numerical order and circle the numbers of the four problems below your name. Within reason, you may quote theorems as long as you state them clearly.

Name: ____________________________________

Problems to be graded: 1 2 3 4 5 6

1. Let F be a field, let $n \geq 2$, and let R be the ring of $n \times n$ matrices over F.
   (a) (5 points) Show that the only 2-sided ideals of R are $\{0\}$ and R.
   (b) (5 points) Show that R contains a nonzero proper left ideal I.

2. Give an example of each of the following, with some explanation.
   (a) (3 points) A UFD which is not a PID.
   (b) (3 points) A ring R with unit and a ring S with unit such that S is a subring of R but $1_R \neq 1_S$.
   (c) (4 points) A commutative ring R with unit and an irreducible element $\pi \in R$ which is not prime.

3. (10 points) Prove that the ideal (X, Y) in the ring $\mathbf{Q}[X, Y]$ is not a principal ideal.

4. Let R be a ring and let M be an R-module. Define

$$\mathrm{Tor}(M) = \{x \in M : rx = 0 \text{ for some } r \in R \smallsetminus \{0\}\}.$$

   (a) (5 points) Prove that if R is an integral domain then $\mathrm{Tor}(M)$ is a submodule of M.
   (b) (5 points) Give an example of a ring R and an R-module M such that $\mathrm{Tor}(M)$ is not a submodule of M. Prove any claims that you make.

5. (10 points) Find a representative for each conjugacy class in $\mathrm{GL}_2(\mathbf{Z}/3\mathbf{Z})$ consisting of elements whose order divides 8. (Note that we have $X^4+1 = (X^2-X-1)(X^2+X-1)$ in $(\mathbf{Z}/3\mathbf{Z})[X]$.)

6. (10 points) Let L/K be an algebraic field extension and let R be a subring of L that contains K. Prove that R is a field.