Department of Mathematics, University of Florida
Second Semester Algebra Exam – May, 2015

Answer four problems. If you turn in more than four, only the first four will be graded. Within reason, you may use theorems as long as you state them clearly. When you are done, put the answers in numerical order, put your name in the space below and circle the numbers of the problems you wish graded.

Name:

Problems: 1   2   3   4   5   6

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1.   (i) Find all units in the ring $R = \mathbb{Z}[\sqrt{-7}]$. (ii) Show that R is not a unique factorization domain. (iii) Show that $(2, 1 + \sqrt{-7})$ is a maximal ideal of R.

2.   Suppose R is a commutative ring and $I \subset R$ is an ideal. Show that the set of $r \in R$ such that $r^n \in I$ for some $n > 0$ is an ideal of R.

3.   Let R be an integral domain and M a finitely generated R-module. We say that M has *rank* n if there are n elements $m_1, \ldots, m_n \in M$ that are linearly independent, while any subset of M with cardinality greater than n is linearly dependent. Show that M has rank n if and only if M has a submodule $N \subseteq M$ such that (1) N is free of rank n, and (2) M/N is torsion.

4.   Which of the following polynomials are irreducible? Explain.

$$(i) \quad X^3 + 5X + 1 \in \mathbb{Q}[X]$$
$$(ii) \quad X^4 + X + 2 \in \mathbb{Q}[X]$$
$$(iii) \quad X^5 + Y^5 - 1 \in \mathbb{C}[X, Y]$$

5.   Suppose M is an $n \times n$ matrix with coefficients in an algebraically closed field. Show that the following statements are equivalent:

1. M is nilpotent, i.e. $M^k = 0$ for some $k \geq 0$.

2. 0 is the only eigenvalue of M.

3. M is similar to an upper triangular matrix with 0s on the diagonal.

6.   Find representatives for each similarity class of matrices in $M_5(\mathbb{C})$ whose minimal polynomial is $X^3 + X^2 - X - 1 = (X + 1)^2(X - 1)$