Second Semester Algebra Exam – January 5, 2015

Answer four problems. If you turn in more than four, only the first four will be graded. Within reason, you may use theorems as long as you state them clearly. When you are done, put the answers in numerical order, put your name in the space below and circle the numbers of the problems you wish graded.

Name:

Problems: **1 2 3 4 5 6**

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1. Suppose R is an integral domain. (i) (5 points) Show that the annihilator of any finitely generated torsion R-module is nonzero. (ii) (5 points) Give an example of a torsion $\mathbb{Z}$-module whose annihilator is zero.

2. Let R be the ring

$$R = \{a + b\sqrt{-2} \,|\, a,\ b \in \mathbb{Z}\} \subseteq \mathbb{C}$$

(you may assume it is a ring). (i) (6 points) Show that R is a unique factorization domain. (ii) (4 points) Find the group of units of R.

3. Let k be a field, V a k-vector space of finite dimension, and $V^* = \mathrm{Hom}(V, k)$ be the dual vector space. Show that $\dim V = \dim V^*$.

4. Suppose R is a commutative ring with identity, and $S \subseteq R$ is a multiplicative subset, i.e. $1 \in S$ and a, $b \in S$ implies $ab \in S$. Show that there is a prime ideal $P \subset R$ such that $P \cap S = \phi$.

5. Which of the following polynomials are irreducible in their respective rings? Explain (5 points each)

$$\begin{aligned} (i) &\quad X^3 - X^2 - X - 2 \in \mathbb{Q}[X] \\ (ii) &\quad Y^3 + X^2Y + X^3Y + X \in k[X, Y],\ k \text{ a field} \end{aligned}$$

6. Let $= GL_3(\mathbb{F}_2)$ be the group of 3×3 invertible matrices with entries in the field with two elements. (i) (5 points) Find representives for all the conjugacy classes of order 2. (ii) (5 points) Do the same for elements of order 7.