Answer **four** problems. (If you turn in more, the first four will be graded.) Put your answers in numerical order and circle the numbers of the four problems below your name. Within reason, you may quote theorems as long as you state them clearly.

Name: ______________________________

Problems to be graded: 1 2 3 4 5 6

1. (10 points) Let $R = \mathbf{Z}[i]$ be the ring of Gaussian integers. Let M be a maximal ideal of R. Prove that R/M is a finite field of order p or p^2 where p is a (rational) prime number.

2. (a) (4 points) Give an example of an integral domain which is not a unique factorization domain.

   (b) (3 points) Give an example of a principal ideal domain which is not a field and is not isomorphic to $\mathbf{Z}$.

   (c) (3 points) Give an example of a commutative ring with identity which is not an integral domain.

3. (a) (3 points) Define *Euclidean* domain.

   (b) (7 points) Prove that every Euclidean domain is a principal ideal domain.

4. (10 points) Let R be a ring with identity and let M be an R-module. Prove that the annihilator $\mathrm{Ann}(M)$ is a two sided ideal of R.

5. Let V be a vector space of dimension 4 over the field $\mathbf{Q}$ of rational numbers. Let $T : V \to V$ be a linear transformation. Suppose that $(T^2 - Id)^2 = 0$.

   (a) (7 points) What are the possible minimal polynomials for T?

   (b) (3 points) For each possible minimal polynomial from (a), give a matrix with this minimal polynomial.

6. (10 points) Let F/K be an extension of fields, and suppose that $[F : K] = 5$. Prove that there exists an irreducible polynomial $p(x) \in K[x]$ such that F is isomorphic to $K[x]/(p(x))$.