Answer **four** problems. (If you turn in more, the first four will be graded.) Put your answers in numerical order and circle the numbers of the four problems below your name. Within reason, you may quote theorems as long as you state them clearly.

Name: ______________________________

Problems to be graded: 1 2 3 4 5 6

1. Give an example of each of the following, with some explanation.
   (a) (3 points) A nonabelian group, all of whose subgroups are normal.
   (b) (3 points) A nonabelian simple group.
   (c) (4 points) A group G and a normal subgroup N of G such that N is not a characteristic subgroup of G.
2. (10 points) Let G be a group. Prove that the following are equivalent:
   (a) G is not equal to the union of its proper subgroups.
   (b) G is a cyclic group.
3. Let S_n denote the symmetric group on n symbols.
   (a) (3 points) Define the sign $\epsilon(\sigma)$ of $\sigma \in S_n$.
   (b) (4 points) Prove that your definition from (a) is independent of any choices that need to be made.
   (c) (3 points) Use your definition from (a) to prove that $\epsilon(\sigma \circ \tau) = \epsilon(\sigma)\epsilon(\tau)$ for all $\sigma, \tau \in S_n$.
4. (10 points) Let G be a nontrivial finite p-group, where p is a prime, and let H be the subgroup of G generated by the set
$$\{[x, y] : x, y \in G\} \cup \{x^p : x \in G\}.$$
   Prove that H is a proper normal subgroup of G.
5. (10 points) Let G be a finite group of order pqr, where p, q and r are primes and $p < q < r$. Prove that G has a normal non-trivial Sylow subgroup.
6. (10 points) For each homomorphism $\phi : \mathbf{Z}/2\mathbf{Z} \to \mathrm{Aut}(\mathbf{Z}/8\mathbf{Z})$ describe the semidirect product $(\mathbf{Z}/8\mathbf{Z}) \rtimes_\phi (\mathbf{Z}/2\mathbf{Z})$. Prove that these groups are pairwise nonisomorphic.