Department of Mathematics, University of Florida
First Semester Algebra Exam – January 5, 2015

Answer four problems. If you turn in more than four, only the first four will be graded. Within reason, you may use theorems as long as you state them clearly. When you are done, put the answers in numerical order, put your name in the space below and circle the numbers of the problems you wish graded.

Name:

Problems: **1 2 3 4 5 6**

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1. Describe the subgroup

$$\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \,\middle|\, ac \neq 0 \right\} \subseteq GL_2(\mathbb{R})$$

as a semidirect product $A \rtimes_\phi B$ with nontrivial A, B. Specify the groups A, B and the homomorphism $\phi : B \to \mathrm{Aut}(A)$.

2. Let G be a p-group and S a finite set on which G acts. Denote by $S^G \subseteq S$ the subset of fixed elements (i.e. $g(x) = x$ for all $g \in G$, $x \in S$. Show that $|S| \equiv |S^G| \pmod{p}$.

3. Let G be a group of order $3^2 \cdot 11 \cdot 41$. (i) (3 points) Show that the 41-Sylow subgroup is normal. (ii) (4 points) Show that in fact it is contained in the center. (iii) (3 points) Show that G has a normal *cyclic* subgroup of order $451 = 11 \cdot 41$.

4. Suppose p is an odd prime. (i) (5 points) Show that if $0 < a < p$ then the p-Sylow subgroup of the symmetric group S_{ap} is an elementary abelian p-group. (ii) (5 points) Show that a p-Sylow subgroup of S_{p^2} is nonabelian of order p^{p+1}.

5. Suppose G is a p-group and $M \subseteq G$ is a maximal subgroup. (i) (6 points) Show that any maximal subgroup of G has index p. (ii) (4 points) Show that M contains all commutators and p-th powers of elements of G.

6. Let G be a finite group and $H \lhd G$ a normal subgroup. Show that G is solvable if and only if H and G/H are solvable.