First-Semester Algebra Exam **May 10, 2013**

Answer four problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. (a) Let G be a group which has 28 cyclic subgroups of order 4. Prove that G has 56 elements of order 4.

   (b) Let n, k be positive integers and let G be a group which has k cyclic subgroups of order n. Determine with proof the number of elements of order n in G.

2. Let G be a group and let H, K be subgroups of G. Recall that $HK = \{xy : x \in H,\ y \in K\}$.

   (a) Prove that HK is a subgroup of G if and only if $HK = KH$.

   (b) Give an example of G, H, K as above such that HK is not a subgroup of G.

3. (a) Let σ, τ be elements of the symmetric group S_n. Prove that σ and τ are conjugate in S_n if and only if they have the same cycle type.

   (b) Give an example of elements σ and τ in the alternating group A_n for some $n \geq 4$ which have the same cycle type but are not conjugate in A_n.

4. Let G be a group of order $2013 = 3 \cdot 11 \cdot 61$. Prove that the center of G contains an element of order 11.

5. Determine a representative for each isomorphism class of abelian groups of order 400.

6. Let $H = \langle x \rangle$ be a cyclic group of order 7 and let $K = \langle y \rangle$ be a cyclic group of order 3.

   (a) Determine all homomorphisms $\phi : K \to \mathrm{Aut}(H)$.

   (b) For each homomorphism ϕ from (a) describe the corresponding semidirect product $H \rtimes_\phi K$.

   (c) Determine with justification which semidirect products from (b) are isomorphic to each other.