First-Semester Algebra Exam **January 14, 2013**

Answer four problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Give an example of each of the following, with some explanation.

   (a) A group G and a subgroup $H \leq G$ such that $C_G(H) \neq N_G(H)$.

   (b) A group G and a subgroup $H \leq G$ such that $H \not\subseteq C_G(H)$.

   (c) A group G and a normal subgroup $H \trianglelefteq G$ such that $C_G(H) \neq G$.

2. Let G be a group and let $T_G = \{x \in G : x^n = 1 \text{ for some } n \geq 1\}$.

   (a) Prove that if G is abelian then T_G is a subgroup of G.

   (b) Give an example of a nonabelian group G such that T_G is not a subgroup of G. Justify any claims that you make.

3. Let G be a finite group which acts on the set S. Let H be a normal subgroup of G with the following property: For every $x_1,\ x_2 \in S$, there is a unique $h \in H$ such that $h(x_1) = x_2$. Choose $x \in S$ and let $G_x = \{g \in G : g(x) = x\}$ denote the stabilizer of x.

   (a) Prove that $G = G_xH$ and $H \cap G_x = 1$.

   (b) Show that if $H \leq Z(G)$ then $G_x \trianglelefteq G$ and G is the internal direct product of G_x and H.

4. (a) Prove that every simple subgroup of S_4 is abelian.

   (b) Use the result from (a) to prove that if G is a nonabelian simple group then every proper subgroup of G has index at least 5.

5. Let G be a group of order $76 = 4 \cdot 19$.

   (a) Prove that G contains a normal Sylow 19-subgroup.

   (b) Prove that the center of G contains an element of order 2.

6. Let G be a group.

(a) Define the derived series $G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \ldots$ of G.

(b) Prove that G is solvable if and only if $G^{(n)} = \{1\}$ for some $n \geq 0$.

(c) Give an example of a solvable group G such that $G^{(1)} \neq \{1\}$.