FIRST-YEAR EXAM - SECOND-SEMESTER TOPOLOGY - SPRING 2019

Answer all questions and work all problems. Make proofs as succinct as possible. Work each problem on a separate sheet of paper. Each problem is worth the points allotted.

Problem 1. (5 points) Let $n \geq 1$. Suppose that $f, g : X \to \mathbb{R}^n$ are continuous. Show that f and g are homotopic.

Problem 2. (5 points) Let $\gamma : [0, 1] \to X$ with basepoint x_0. Let γ^{-1} be defined by $\gamma^{-1}(t) = \gamma(1 - t)$. Show that $\gamma * \gamma^{-1}$ is loop homotopic to the constant loop $1 : [0, 1] \to X$, $1(t) \equiv x_0$.

Problem 3. (5 points) Show that $\pi_1(S^1, 1) = \mathbb{Z}$.

Problem 4. (5 points) Show that the disk, $D^2 = \{z \in \mathbb{C} \mid ||z|| \leq 1\}$ has the fixed point property.

Problem 5. (5 points) Consider the matrix $M = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$. This represents a group homomorphism $M : Z^2 \to Z^2$. Show that there is a map $f : \mathbb{T}^2 \to \mathbb{T}^2$ such that $f_* = M$ where $f_* : \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{T}^2)$ is the homomorphism induced by f on the fundamental group.

Problem 6. (5 points) Suppose that $f : \mathbb{S}^1 \to \mathbb{S}^1$ is continuous. Show that f is homotopic to $z^n : \mathbb{S}^1 \to \mathbb{S}^1$ for some $n \in \mathbb{Z}$. This is the **degree** of f.

Problem 7. (5 points) Let $n > 1$. Let $\mathbb{P}^n = \mathbb{S}^n / D$ where D identifies x with its antipode for all $x \in \mathbb{S}^n$. Show that $\pi_1(\mathbb{P}^n, x_0) \cong \mathbb{Z}_2$.

Problem 8. (5 points) Describe space X such that $\pi_1(X, x_0) \cong \mathbb{Z}_3$. Verify that this is the fundamental group by the **Seifert−van Kampen Theorem**.

Problem 9. (5 points) A topological space is said to be **Lindelöf** provided that for every open cover $\mathcal{U}$ of X, there is a countable $\mathcal{V} \subset \mathcal{U}$ covering X. Show that if X is a separable metric space, then X is Lindelöf.

Problem 10. (5 points) State and prove the **Tychonoff Theorem**.

Problem 11. (50 points) State the following theorems or verify the following statements with quick examples or statements.

The Alexander Subbase Theorem.

The Hahn−Mazurkiewicz Theorem

The Brouwer Characterization of the Cantor Set

The Jordan Curve Theorem

The Borsuk-Ulam Theorem for $\mathbb{S}^2$.

Determine the fundamental group of $\mathbb{T}^2$.

Determine the fundamental group of the connected sum of two tori, $\pi_1\left(\mathbb{T}^2 \# \mathbb{T}^2\right)$.

The Arcwise Connectedness Theorem

Suppose that X is a compact metric AR. Show that for any Z and any pair of continuous functions $f, g : Z \to X$, f and g are homotopic.

Suppose that C is the Cantor set. Show that C is homeomorphic to $C \times C$.