Numerical Linear Algebra Exam: August, 2019
Do **4** (four) problems.

1. Let $A \in \mathbb{C}^{m\times n}$.

(a) Prove or give a counterexample: $\|A\|_2 \le \sqrt{\|A\|_\infty \|A\|_1}$. If you prove this, make sure to justify each nontrivial step.

(b) Prove or give a counterexample: $\|A\|_2 \le \|A\|_F$, where $\|A\|_F$ is the Frobenius norm of A. If you prove this, make sure to justify each nontrivial step.

2. **(a)** Prove that the inverse of an upper-triangular matrix is upper-triangular.

(b) Let $A \in \mathbb{C}^{m\times n}$ with $m > n$. Consider the least-squares problem of finding $x \in \mathbb{C}^n$ that minimizes $\|Ax - b\|$ in the 2-norm. Describe a method for solving the problem efficiently, and explain (and justify) why the normal equations should not solved.

3. Let $A \in \mathbb{C}^{m\times n}$, with $m \ge n$ and $\operatorname{rank}(A) = p = n \ge 3$. Let $a_1, a_2, \ldots$ denote the columns of A.

(a) Using the classical Gramm-Schmidt process, write out expressions for q_1, q_2, q_3, the first three columns of Q in the QR decomposition of A.

(b) Show the vector q_3 found in part **(a)** is orthogonal to both q_1 and q_2.

(c) Write an expression for the first Householder reflector H_1, used to find the QR decomposition of A. Show H_1 is both unitary and Hermetian.

4. Let $A \in \mathbb{C}^{m\times m}$ be Hermetian.

(a) Show that all eigenvalues of A are real.

(b) Define the stationary iterative method (a.k.a. fixed point method)

$$x^{(k+1)} = Ax^{(k)} + b. \tag{1}$$

Suppose (1) has fixed-point x, namely x satisfies $x = Ax + b$. Show the iteration (1) converges to x from any starting guess $x^{(0)}$, that is $x^{(k)} \to x$ as $k \to \infty$, if and only if the eigenvalues λ_i of A satisfy $|\lambda_i| < 1,\ i = 1, \ldots, m$. You may use the fact that Hermetian matrix A is unitarily diagonalizable.

5. Consider the matrix A given by

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ -1 & 4 & -1 & 1 \\ 2 & -1 & 6 & -2 \\ 0 & 1 & -2 & 4 \end{pmatrix}$$

Suppose the eigenvalues of A are all distinct (they are) and satisfy $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$.

(a) Show that A is positive definite.

(b) Describe an algorithm that could be used to approximate λ_4.

(c) Describe algorithms that could be used to approximate λ_2, λ_3, and their eigenvectors.