Numerical Linear Algebra Exam: January, 2019
Do **4** (four) problems.

1. Let $A = U\Sigma V^*$ be the singular value decomposition of $A \in \mathbb{C}^{m\times n}$ with $\operatorname{rank}(A) = p$ and $p \le n \le m$.

- **(a)** Show $\operatorname{Col}(A) = \operatorname{Span}\{u_1, u_2, \dots, u_p\}$, where $u_1, \dots, u_p$ are the first p columns of U.
- **(b)** Show $\operatorname{Null}(A^*) = \operatorname{Span}\{u_{p+1}, u_{p+2}, \dots, u_m\}$.
- **(c)** Show that A^*A is invertible if and only if A is full rank.

2. Let matrix $A \in \mathbb{C}^{m\times n}$, with $n < m$. Let vector $b \in \mathbb{C}^m$, and let r denote the residual vector $r = b - Ax$.

- **(a)** Show that x solves the least-squares problem $\min \|b - Ax\|_2$ if and only if $r \in \operatorname{Null}(A^*)$.
- **(b)** Suppose A is full rank, and describe how to find the least-squares solution using the QR decomposition of A.

3.
- **(a)** Show that if $A \in \mathbb{C}^{m\times m}$, A has a Schur decomposition.
- **(b)** Show that if $T \in \mathbb{C}^{m\times m}$ is normal and triangular, then T is diagonal.
- **(c)** Show that if $A \in \mathbb{C}^{m\times m}$ is normal and λ is an eigenvalue of A, then the geometric multiplicity of λ is equal to the algebraic multiplicity of λ.

4. Let $\|\cdot\|$ be a subordinate (induced) matrix norm. If A is $n \times n$ invertible and E is $n \times n$ with $\|A^{-1}\|\|E\| < 1$, then show

- **(a)** $A + E$ is nonsinguar
- **(b)**

$$\|(A+E)^{-1}\| \le \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|E\|}.$$

5. Consider the matrix A given by

$$A = \begin{pmatrix} 1 & 1 & 2 & -3 \\ 1 & 10 & -1 & 1 \\ 2 & -1 & 13 & -2 \\ -3 & 1 & -2 & 25 \end{pmatrix}$$

- **(a)** Show that A is positive definite.
- **(b)** What can you say about the location of each of the eigenvalues of A? Your answer should be in the form of an interval or a union of intervals.
- **(c)** Suppose the eigenvalues of A are all distinct (they are) and satisfy $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$. Describe an algorithm that could be assured to converge to λ_4.