MAA 5229 First-Year Exam, August 2019

Do exactly **4** problems. Work must be presented in a neat and logical fashion in order to receive credit. Do not leave any gaps. When a theorem is used in a proof it must be precisely stated.

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1. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function and suppose that $\int_a^b f(x)\,dx > 0$. Prove that there is a nonempty open interval $I \subset [a, b]$ and a number $\delta > 0$ such that $f(x) > \delta$ for all $x \in I$. Does this remain true if f is only assumed to be Lebesgue integrable? Prove, or give a counterexample.

2. Let (f_n) be a sequence of continuously differentiable functions on $[0, 1]$. Prove that if $f_n' \to g$ uniformly on $[0, 1]$, and the sequence $f_n(0)$ converges, then there is a continuously differentiable function f such that $f_n \to f$ uniformly on $[0, 1]$ and $f' = g$.

3. Let f be continuous on $[0, a]$ and suppose that
$$\int_0^a e^{-st} f(t)\,dt = 0$$
for all $s \geq 0$. Prove that $f = 0$.

4. Let $(X, \mathscr{M})$ be a measurable space and $f_n : X \to \mathbb{R}$ a sequence of measurable functions. Prove that each of the following sets is measurable:
   a) the set of points $x \in X$ at which $f_n(x)$ is rational for all n
   b) the set of points $x \in X$ for which the sequence $f_n(x)$ is decreasing
   c) the set of points $x \in X$ at which the sequence $(f_n(x))$ diverges

5. State the Monotone Convergence Theorem and Fatou's theorem, and use the former to prove the latter.