MAA 5229 First-Year Exam, May 2019

Do exactly **4** problems. Work must be presented in a neat and logical fashion in order to receive credit. Do not leave any gaps. When a theorem is used in a proof it must be precisely stated.

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1. Let $f_n : [a,b] \to \mathbb{R}$ be a uniformly bounded sequence of Riemann integrable functions. Prove that if $f_n \to f$ uniformly, then f is Riemann integrable.

2. Let $f : [0,1] \to \mathbb{R}$ be continuous. Suppose that

$$\int_0^1 f(x) x^{2019k}\, dx = 0$$

for all integers $k \geq 0$. Must f be identically 0? Prove, or give a counterexample.

3. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of measurable functions and suppose there is a function $g \in L^1[0,1]$ such that $|f_n| \leq g$ for all n. Consider the functions

$$F_n(t) = \int_0^t f_n(x)\, dx.$$

Prove i) each F_n is continuous on $[0,1]$ and ii) there is a subsequence (F_{n_k}) converging uniformly on $[0,1]$.

4. Let $(X, \mathscr{M})$ be a measurable space and $f_n : X \to \mathbb{R}$, $n = 1, 2, 3, \ldots$ a sequence of measurable functions. Prove that each of the following subsets of X is measurable:

   a) $\{x | f_n(x) > 0 \text{ for infinitely many values of } n\}$
   b) $\{x | \text{ the sequence } (f_n(x)) \text{ is eventually monotone}\}$
   c) $\{x | \lim_{n\to\infty} n f_n(x) = 0\}$

5. Let $f_1 \geq f_2 \geq f_3 \geq \cdots$ be nonnegative measurable functions on a measure space $(X, \mathscr{M}, \mu)$, and put $f = \lim f_n$. Suppose that $\int f_k\, d\mu < \infty$ for some k. Prove that

$$\int f\, d\mu = \lim \int f_n\, d\mu.$$

Give a counterexample to show that the conclusion can fail if the finiteness hypothesis is dropped.