MAA 5228 First-Year Exam, August 2019

Do exactly **4** problems. Work must be presented in a neat and logical fashion in order to receive credit. Do not leave any gaps. When a theorem is used in a proof it must be precisely stated.

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1. a) Let X be a metric space, (p_n) a sequence in X, and $p \in X$. Prove that (p_n) converges to p if and only if every subsequence of (p_n) has a subsequence converging to p.

   b) Give an example of a metric space X and a sequence (p_n) in X such that (p_n) diverges, but every subsequence of (p_n) has a convergent subsequence.

2. Let $f : (0, 1) \to \mathbb{R}$ be a uniformly continuous function. Prove that $\lim_{x \to 0^+} f(x)$ exists. Does the conclusion remain true if f is only assumed bounded and continuous?

3. Let X be a metric space and S a connected subset of X. Must $\overline{S}$ (the closure of S) be connected? Prove, or give a counterexample. If S° (the interior of S) is nonempty, must it be connected? Prove, or give a counterexample.

4. Let X and Y be disjoint, nonempty subsets of $\mathbb{R}^n$, with X closed and Y compact. Define

$$dist(X, Y) := \inf_{x \in X, y \in Y} \|x - y\|.$$

   Prove that there exist points $x_0 \in X$, $y_0 \in Y$ with $\|x_0 - y_0\| = dist(X, Y)$.

5. Let $f : [0, a) \to \mathbb{R}$ be differentiable, and suppose that f' is strictly increasing on $[0, a)$. Prove that the function

$$g(x) = \begin{cases} f'(0) & x = 0 \\ \frac{f(x)}{x} & 0 < x < a \end{cases}$$

   is continuous and strictly increasing on $[0, a)$.