MAA 5228 First-Year Exam, May 2019

Do exactly **4** problems. Work must be presented in a neat and logical fashion in order to receive credit. Do not leave any gaps. When a theorem is used in a proof it must be precisely stated.

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1. Let (a_n) and (b_n) be bounded real sequences, with $\lim a_n = a$. Prove that

$$\limsup(a_n + b_n) = a + \limsup b_n.$$

2. Let (X, d) be a metric space and $E \subseteq X$ a nonempty subset. Consider the function $d_E : X \to [0, +\infty)$ defined by

$$d_E(x) = \inf_{y \in E} d(x, y).$$

Prove i) d_E is uniformly continuous on X, and ii) describe, with proof, the set of points at which $d_E = 0$.

3. Let X be a compact metric space. Let $\{F_\alpha\}_{\alpha \in A}$ be a (nonempty) collection of nonempty closed subsets of X, and suppose that $\{F_\alpha\}$ is *totally ordered* by inclusion (this means that for every $\alpha, \beta \in A$, either $F_\alpha \subseteq F_\beta$ or $F_\beta \subseteq F_\alpha$). Prove that the intersection $\cap_{\alpha \in A} F_\alpha$ is nonempty.

4. Let $\{U_\alpha\}_{\alpha \in A}$ be a family of connected subsets of a metric space X. Suppose that $U_\alpha \cap U_\beta \neq \varnothing$ for each pair $\alpha, \beta \in A$. Prove that $\cup_{\alpha \in A} U_\alpha$ is connected.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, and suppose that f' is monotone. Must f' be continuous? Prove, or give a counterexample.