MAA 5228 First-Year Exam, January 2019

Do exactly **4** problems. Work must be presented in a neat and logical fashion in order to receive credit. Do not leave any gaps. When a theorem is used in a proof it must be precisely stated.

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1. Let $(s_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. Consider the sequence (σ_n) of averages
$$\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$
Prove that the sequence (σ_n) is bounded and
$$\limsup \sigma_n \leq \limsup s_n.$$

2. Let $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ be a decreasing family of nonempty sets in a metric space X.
   a) Prove that if each K_n is compact, then $\cap_n K_n$ is nonempty.
   b) Given an example to show that $\cap_n K_n$ can be empty if the K_n are only assumed closed.

3. Let X, Y be metric spaces and $f : X \to Y$ a function. Prove that if f is uniformly continuous on X and (x_n) is a Cauchy sequence in X, then $(f(x_n))$ is a Cauchy sequence in Y.

4. Let X be a metric space and suppose that every continuous function $f : X \to \mathbb{Z}$ is constant. Prove that X is connected.

5. Suppose f is differentiable in (a, b) and $f'(x) > 0$ for all $x \in (a, b)$. Let g be the inverse function to f. Prove that for each $x_0 \in (a, b)$, the function g is differentiable at $f(x_0)$ and
$$g'(f(x_0)) = \frac{1}{f'(x_0)}.$$