Answer **four** problems. (If you turn in more, the first four will be graded.) Put your answers in numerical order and circle the numbers of the four problems below your name. Within reason, you may quote theorems as long as you state them clearly.

Name: ______________________________

Problems to be graded: 1 2 3 4 5 6

1. (10 points) State and prove the Chinese Remainder Theorem for rings.

2. Let $R = \mathbf{Z}\left[\sqrt{-1}\right]$ be the ring of Gaussian integers. Suppose that F is a finite field of characteristic 5, and $\phi\colon R \to F$ is a (unital) ring homomorphism. Prove that $R/\ker(\phi)$ has exactly 5 elements.

3. Let $\omega = \frac{-1+\sqrt{-3}}{2} \in \mathbf{C}$.
   (a) (3 points) Prove that ω is a root of the polynomial $f(x) = x^2 + x + 1$.
   (b) (3 points) Prove that the set $\mathbf{Z}[\omega] = \{a + b\omega : a, b \in \mathbf{Z}\}$ is a subring of $\mathbf{C}$.
   (c) (4 points) Prove that $\mathbf{Z}[\omega]$ is a Euclidean domain with respect to the norm $N(\alpha) = \alpha\overline{\alpha}$ for all $\alpha \in \mathbf{Z}[\omega]$, where $\overline{\alpha}$ denotes the complex conjugate of α.

4. Let R be an integral domain, and M be an R-module.
   (a) (3 points) Define the *rank* of M;
   (b) (7 points) Prove that if M is a free R-module of rank n, (where n is a non-negative integer), then the rank of any submodule of M is at most n.

5. (a) (7 points) Let p be an odd prime. Let $A \in \mathrm{GL}(2, p)$ be an invertible two by two matrix over the field with p elements such that A has order 2 and $A \neq -I$ (where I is the identity matrix). Prove that A is conjugate to the diagonal matrix whose diagonal entries are -1 and 1.
   (b) (3 points) Give an example of an element of order 2 in $\mathrm{GL}(2, 2)$.

6. Let F/K be a field extension.
   (a) (2 points) Define what it means to say that $\theta \in F$ is *algebraic* over K.
   (b) (4 points) Let A be the set of all elements of F that are algebraic over K. Prove that A is a subfield of F.
   (c) (4 points) Prove that every element of F that is not in A is transcendental over A.