Department of Mathematics, University of Florida
Second Semester Algebra Exam – May, 2017

Answer four problems. If you turn in more than four, only the first four will be graded. Within reason, you may use theorems as long as you state them clearly. When you are done, put the answers in numerical order, put your name in the space below and circle the numbers of the problems you wish graded.

Name:

Problems: **1** **2** **3** **4** **5** **6**

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1. Let $\omega = \frac{-1+\sqrt{-3}}{2}$. Show that the ring $\mathbb{Z}[\omega]$ of all complex numbers of the form $a + b\omega$ with $a, b \in \mathbb{Z}$ is a Euclidean domain.

2. Let F be a field (i) (6 points) Show that if F has characteristic zero and $n \geq 1$ is an integer, the ring $F[X, Y]/(X^n + Y^n - 1)$ is an integral domain. (Hint: first show that $Y - 1$ and $Y^{n-1} + Y^{n-2} + \cdots + Y + 1$ are relatively prime in $F[Y]$). (ii) (4 points) Find an example of a field of positive characteristic and $n \geq 1$ for which the conclusion of (i) is false.

3. Let R be an integral domain. Recall that an R-module M has *rank* n if n is the largest integer such that M has n R-linearly independent elements. Show that an R-module M has rank n if and only if M has a submodule $N \subseteq M$ such that N is free of rank n, and M/N is torsion.

4. Let F be a field, M an extension of F and L, K two finite extensions of F contained in M. Show that if $[L : F]$ and $[K : F]$ are relatively prime, $L \cap K = F$.

5. Show that the group $GL_3(\mathbb{F}_2)$ has two distinct conjugacy classes of elements of order 7, and find representatives for each class.

6. Let F be a field, V an F-vector space of finite dimension and $T : V \to V$ a linear transformation. Show that if the minimal polynomial of T is irreducible of degree d then the dimension of V is a multiple of d.