Department of Mathematics, University of Florida
First Semester Algebra Exam – May, 2017

Answer four problems. If you turn in more than four, only the first four will be graded. Unless otherwise indicated you may use theorems as long as you state them clearly. When you are done, put the answers in numerical order, put your name in the space below and circle the numbers of the problems you wish graded.

Name:

Problems: **1** **2** **3** **4** **5** **6**

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1. Let p be a prime and G be a finite group whose order is a power of p greater than 1. (i) Show that the center of G is not trivial. (ii) Deduce that G is solvable. (Do not just state theorems; give the proof).

2. Show that a group of order $132 = 2^2 \cdot 3 \cdot 11$ has a normal Sylow subgroup.

3. Let G be the group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| \, a,\ b, c \in \mathbb{R},\ ac \neq 0 \right\}$$

and let H (resp. N) be the subgroup of G consisting of elements in which $b = 0$ (resp. $a = c = 1$). Show that G is isomorphic to a semidirect product of H and N, and identify the relevant map ϕ from one to the automorphism group of the other.

4. Let G be a group of permutations of a set A, and for $a \in A$ let G_a be the stabilizer of a. (i) Show that $gG_ag^{-1} = G_{g(a)}$ for all $g \in G$, $a \in A$. (ii) Show that if G is transitive then

$$\bigcap_{g \in G} gG_ag^{-1} = 1.$$

(iii) Show that if G is abelian and transitive then $g \neq 1$ implies $g(a) \neq a$ for all $a \in A$. Conclude that if A is finite then so is G, and $|G| = |A|$.

5. Let p and q be distinct primes. List all abelian groups of order p^3q^2. For each such group, how many elements are there of order p? pq?

6. (i) List representatives of all conjugacy classes in S_9 whose elements have order 3. (ii) Find the cardinality of every such class.