University of Florida

MAD6407 EXAM May 9, 2014

Name:

ID #:

Instructor: Maia Martcheva

**Directions:** You have 2 hours to answer the following questions. You must show all your work as neatly and clearly as possible and indicate the final answer clearly. You may not use a calculator.

| Problem | Possible | Points |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 20 | |
| Total | 100 | |

(1) (20 points) Show that the fixed point equation $x = f(x)$ with a fixed point $\alpha$ can be solved iteratively, if $f'(\alpha) \neq 1$, by one of the following fixed point iteration formulas:

(a) $x_{n+1} = f(x_n)$

(b) $x_{n+1} = f^{-1}(x_n)$.

Hint: The following formula for the derivative of an inverse is valid

$$[f^{-1}]'(x) = \frac{1}{f'(f^{-1}(x))}$$

(c) $x_{n+1} = (x_n + f(x_n))/2$

(2) (20 points) Let $f(x) \in C[a, b]$. Let $p(x)$ be a polynomial for which

$$||f' - p||_\infty \leq \epsilon$$

and define

$$q(x) = f(a) + \int_0^x p(t)dt, \qquad a \leq x \leq b.$$

Show that $q(x)$ is a polynomial that satisfies

$$||f - q||_\infty \leq \epsilon(b - a)$$

(3) (20 points) Consider the integral

(1) $$\int_0^\pi x^2 \cos x \, dx$$

(a) Consider a quadrature rule of the form

$$\int_0^\pi x^\alpha f(x) \, dx \approx A f(0) + B \int_0^\pi f(x) \, dx$$

where $\alpha > -1, \alpha \neq 0$ is a parameter. Determine the constants $A, B$ so that the quadrature formula has degree of exactness one.

(b) Use the formula in part (a) to approximate the integral (1).

(4) (20 points) For the function $f(x) = \ln(1 + x)$ for $x \in [0, 1]$, find the minimax approximation polynomial of degree one. Give the exact value of the minimax error.

(5) (20 points) Assume that you are solving the initial value problem

$$\begin{aligned} y' &= f(t,y) \qquad\qquad a \le t \le b \\ y(a) &= \alpha \end{aligned}$$

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(a) Derive the formula for the global error of thenumerical solutions for the ODE problem above obtained via Euler's method.
Hint: The formula is ($M = ||Y''||_\infty$):

$$|Y(t_i) - w_i| < \frac{hM}{2L}[e^{L(b-a)} - 1].$$

(b) Compute the value of $M = ||Y''||_\infty$ necessary to apply the global error formula above to the specific ODE problem

$$\begin{aligned} y' &= \sin(t + 2y) + e^t \qquad\qquad 0 \le t \le 1 \\ y(0) &= 0 \end{aligned}$$