Numerical Analysis Prelim, May 10, 2013
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1. (General Fixed Point Theory) Assume that $g(x)$ is continuously differentiable on $[a, b]$, and that $g([a, b]) \in [a, b]$, and that
$$\lambda = \max_{a \le x \le b} |g'(x)| < 1.$$
Show that the following are true:
(i) $x = g(x)$ has a unique solution α in $[a, b]$,
(ii) For any initial choice x_0 in $[a, b]$, with $x_{n+1} = g(x_n)$, $\lim_{n\to\infty} x_n = \alpha$, and
(iii)
$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha).$$
(iv) Does the statement remain true if the interval (a, b) is open?
(v) Assume in addition that $g'(\alpha) = 0$ and show that the sequence converges quadratically.
(vi) (Newton's Method) Assume that $f(x)$, $f'(x)$, and $f''(x)$ are continuous in $[a, b]$, and that for some $\alpha \in [a, b]$, $f(\alpha) = 0$, and $f'(\alpha) \neq 0$. Then if x_0 is chosen close enough to α, the iterates
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
converge to α. Moreover they converge quadratically.
(vii) State Newton's method for $f(x) = 0$ if $f : \mathbb{R}^n \to \mathbb{R}^n$.

2. Theorem: (Lagrange Error Formula) Let $x_0, x_1, x_2, \ldots, x_n$ be distinct real numbers, $l_j(x)$ be the corresponding Lagrange polynomials and suppose that f is a given real-valued function with $n + 1$ continuous derivatives. Let I be an interval containing all of the x_k, and t. Then there is a $\xi \in I$ such that
$$f(t) - \sum_{j=0}^{n} f(x_j) l_j(t) = \frac{(t - x_0)(t - x_1)...(t - x_n)}{(n + 1)!} f^{(n+1)}(\xi).$$

3. Let $\{p_n\}$ be an orthogonal family on $[a, b]$ constructed by using the Gram-Schmidt process on $1, t, t^2, t^3$...... Prove that all of the zeros of $p_n(t)$ are contained in $[a, b]$.

4. Given Simpson's rule for numerical integration, i.e.
$$\int_{t_n}^{t_{n+2}} f(x)dx \approx \frac{2h}{6}(f(t_n) + 4f(t_{n+1}) + f(t_{n+2})), \quad t_n = t_0 + nh,$$
(i) Explain where the formula comes from (you don't need to derive it)
(ii) Prove that it is exact for cubic polynomials
(iii) Show that the local error is $O(h^5)$.

5. Gaussian Quadrature: Show that you can find n points on an interval $[a, b]$ (tell us which points), and a formula which uses only the value of the function $f(x)$ at these points, and weights w_k such that the approximation formula
$$\int_a^b f(x)dx \approx I_n(f) \equiv \sum_{k=1}^{n} w_k f(x_k)$$
is exact for polynomials of order $2n - 1$.

6. Assume that you are solving the initial value problem $y'(t) = f(t, y)$, with $y(0)$ given. Assume further that $f(t, y)$ satisfies the Lipschitz condition $|f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2|$ for all $t \in [a, b]$. Explain Euler's method and derive a cumulative error formula, not just a local error formula.

7. (General Multistep Methods) Assume that you are solving the initial value problem $y'(t) = f(t, y)$, with $y(0)$ given. Consider a general formula of the type

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h \sum_{j=-1}^{p} b_j f(x_{n-j}, y_{n-j}).$$

Furthermore, let us define the local truncation error as

$$T_n(Y) = Y(t_{n+1}) - \left(\sum_{j=0}^{p} a_j Y(t_{n-j}) + h \sum_{j=-1}^{p} b_j f(t_{n-j}, Y(t_{n-j})) \right),$$

where $Y(t_n)$ is the exact value of the initial value problem and y_n is the approximated value of the problem at t_n. We let

$$\tau_n(Y) = \frac{1}{h} T_n(Y).$$

Given this prove the following

Let $m \geq 1$ be a given integer. In order that $\max |\tau(Y)| \to 0$ as $h \to 0$ for all continuously differentiable $Y(x)$, it is necessary and sufficient that

$$\sum_{j=0}^{p} a_j = 1, \quad \text{and} \quad -\sum_{j=0}^{p} j a_j + \sum_{j=-1}^{p} b_j = 1 \tag{1}$$

Furthermore, for $\tau(h) = O(h^m)$ for functions $Y(x)$ that are $m + 1$ times continuously differentiable, it is necessary and sufficient that (**??**) hold and

$$\sum_{j=0}^{p} (-j)^i a_j + i \sum_{j=-1}^{p} (-j)^{i-1} b_j = 1, \quad \text{for} \quad i = 2, 3, \ldots m.$$

Appendix: Recall that the Hermite polynomials can be written as

$$H_n(x) = \sum_{j=1}^{n} f(x_j) h_j(x) + \sum_{j=1}^{n} f'(x_j) \tilde{h}_j(x).$$

If the points x_j are chosen to be the zeros of the orthogonal polynomials on $[a, b]$, then

$$h_j(x) = \frac{\psi_n(x) l_j(x)}{\psi_n'(x_j)},$$

where $l_j(x)$ is the Lagrange interpolant for x_j.