Numerical Analysis Qualifying Exam (May, 2011). *Answer any 8 questions.*

1. Suppose A and B are square matrices such that AB is normal. Prove that $\|AB\|_2 \le \|BA\|$. (We use $\|\cdot\|_2$ to denote the spectral norm and $\|\cdot\|$ to denote any induced matrix norm.)

2. Let P and Q be Hermitian positive definite matrices. Prove that
$$x^* P x \le x^* Q x \text{ for all } x \in \mathbb{C}^n$$
if and only if
$$x^* Q^{-1} x \le x^* P^{-1} x \text{ for all } x \in \mathbb{C}^n.$$

3. Suppose A is a Hermitian positive definite matrix split into $A = C + C^* + D$ where D is also Hermitian positive definite. Prove that $B = C + \omega^{-1} D$ is invertible whenever $0 < \omega < 2$. Consider the iteration $x_{n+1} = x_n + B^{-1}(b - Ax_n)$, with any initial iterate x_0. Prove that x_n converges to $x = A^{-1}b$ whenever $0 < \omega < 2$.

4. Let the singular values of any matrix $M \in \mathbb{C}^{m\times n}$ be denoted by $\sigma_1(M) \ge \sigma_2(M) \ge \ldots \ge \sigma_q(M)$ with $q = \min(m, n)$. Prove that if A and B are two matrices in $\mathbb{C}^{m\times n}$, then
$$\sigma_{i+j-1}(A + B) \le \sigma_i(A) + \sigma_j(B)$$
for all $i, j = 1, 2, \ldots, q$ and $i + j \le q$.

5. Let $A \in \mathbb{R}^{N\times N}$ and $b \in \mathbb{R}^N$. Consider the following iteration for solving $Ax = b$, that computes x_{n+1}, given $x_n \in \mathbb{R}^N$, as follows (in m intermediate steps): Setting $x_{n+1}^{(0)} = x_n$, for $\ell = 1, 2, \ldots, m$, compute $x_{n+1}^{(\ell)} = x_{n+1}^{(\ell-1)} + \tau_\ell(b - Ax_{n+1}^{(\ell-1)})$. Then, define $x_{n+1} = x_{n+1}^{(m)}$. There is a linear operator E such that $x_{n+1} - x = E(x_n - x)$. Give a formula for E. Suppose A is Hermitian and positive definite with spectral condition number κ. Prove that there are real values of the m parameters τ_ℓ such that
$$\rho(E) \le 2\left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^m.$$

6. Let $I_n(f) = \sum_{j=0}^n w_{j,n} f(x_{j,n})$ be a sequence of quadratures on $[a, b]$ such that (i) $I_n(p) \to I(p)$ as $n \to +\infty$ for any polynomial $p(x)$, and (ii) $B = \sup_n \sum_{j=0}^n |w_{j,n}| < +\infty$. Prove that $I_n(f) \to I(f)$ for all $f \in C[a, b]$. Here, the notation $I(f) = \int_a^b f(x)\,dx$ is used.

7. Design a stable quadratic algorithm for computing the positive root α of the equation $x^2 + 2px - q = 0$ where p and q are arbitrary positive numbers. Prove that your algorithm is indeed stable and quadratic. Find an interval $[a, b]$ such that any iterative sequence starting in $[a, b]$ will converge to α.

8. State the Lagrange interpolation problem. Prove that the problem is well-posed, i.e. prove the existence and uniqueness of solutions. Derive the error formula for the interpolating polynomial.

9. Describe the Trapezoidal Rule for numerical integration. State and prove the error formula of the Trapezoidal Rule. Show that the error bound cannot be improved.

10. Let $w(x) > 0$ be integrable on $[a, b]$. Define an orthogonal polynomial family $\{\phi_n\}$ on $[a, b]$ with weight $w(x)$. Explain how $\{\phi_n\}$ can be constructed from $\{x^n\}$. Prove that $\phi_n(x)$ has n distinct roots in the interval (a, b).