Numerical Analysis Qualifying Examination

September 2005

Do any eight of the ten problems below.

1. For solving $dy/dt = f(t,y)$, consider a one-step method of the form

$$y_1 = y_0 + h\Phi(t_0, y_0, h).$$

For any t_0, it computes an approximation y_1 to $y(t_0+h)$, given $y(t_0) = y_0$. Assuming enough smoothness on f, study the local truncation error of Runge-Kutta methods for which

$$\begin{aligned} \Phi(t_0, y_0, h) &= \alpha_1 k_1 + \alpha_2 k_2, && \text{where} \\ k_1 &= f(t_0, y_0), && \text{and} \\ k_2 &= f(t_0 + \mu h, y_0 + \mu h k_1). \end{aligned}$$

Under what conditions on the constants α_1, α_2, and μ do we get second order methods?

2. Prove that any polynomial q of degree at most $n-1$ satisfies

$$\sum_{i=0}^{n} q(x_i) \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{1}{x_i - x_j} = 0,$$

for any distinct $n+1$ numbers x_i, $i = 0, 1, \ldots, n$. (Suggestion: Identify the left hand side as a divided difference.)

3. Let $p_2(x)$ be the quadratic polynomial that interpolates $f(x)$ at nodes $x = 0, h$, and $2h$.

   (a) Approximating

   $$Q = \int_0^{3h} f(x)\,dx \quad \text{by} \quad Q_h = \int_0^{3h} p_2(x)\,dx,$$

   derive a quadrature rule: Explicitly calculate w_1, w_2, w_3 so that

   $$Q_h = w_1 f(0) + w_2 f(h) + w_3 f(2h).$$

   (b) Assuming that $f(x)$ is four times continuously differentiable, show that

   $$Q - Q_h = \frac{3}{8} h^4 f'''(0) + O(h^5).$$

4. Denote the complete (or clamped) and natural cubic splines interpolating $f \in C^2[a,b]$ at knots $t_0 < t_1 < \cdots < t_n$ by $s_c(t)$ and $s_n(t)$, respectively.

   (a) Prove that

   $$\int_a^b |s_c''(x)|^2\,dx \le \int_a^b |f''(x)|^2\,dx.$$

   State an analogous inequality for s_n.

   (b) Which of the two interpolating splines has smaller "energy", i.e., which of $\int_a^b |s_c''(x)|^2\,dx$ and $\int_a^b |s_n''(x)|^2\,dx$ is smaller?

5. Let $p > 0$ and

$$x = \sqrt{p + \sqrt{p + \sqrt{p + \cdots}}}$$

where all the square roots are positive. Let $F(y) = \sqrt{p+y}$. (How is the fixed point of F related to x?)

(a) Consider the fixed point iteration $x_{n+1} = F(x_n)$. Prove that if the initial iterate x_0 satisfies $x_0 + p > 0$ then all iterates remain on the same side of x as x_0.

(b) Prove that the fixed point iteration converges for all choices of initial guesses greater than $-p + 1/4$.

(c) Whenever the fixed point iteration converges, what is its order of convergence?

6. (a) Consider the matrix $A = uv^*$ where u and $v \in \mathbb{C}^n$. Under what condition on u and v is A a projector?

(b) Show that the Householder matrix $H = I - 2ww^*$ where $\|w\| = 1$ is unitary.

(c) Given a vector $x \in \mathbb{C}^n$ and an integer k with $1 < k < n$, derive a formula for a Householder matrix with the property that $(Hx)_i = 0$ for $i > k$ and $(Hx)_i = x_i$ for $i < k$. Be sure to choose the signs so that the formula is numerically stable.

7. Consider the conjugate gradient algorithm for solving $Ax = b$ for a symmetric positive definite $A \in \mathbb{R}^{m\times m}$.

(a) State a relationship between the n-th iterate of the algorithm and the best approximation to x from a Krylov space (be sure to specify the norm).

(b) Prove that if A has only $n \le m$ distinct eigenvalues then the iteration (in the absence of round off errors) converges to the exact solution x in at most n steps no matter what the initial iterate is. (You may use 7a.)

8. Let P and Q be two $m \times m$ orthogonal projectors.

(a) Prove that $\|P - Q\|_2 \le 1$.

(b) Prove that if $\|P - Q\|_2 < 1$ then the ranges of P and Q have equal dimensions.

9. Let P and Q be Hermitian positive definite matrices. Prove that

$$x^*Px \le x^*Qx, \quad \text{for all } x \in \mathbb{C}^n,$$

if and only if

$$x^*Q^{-1}x \le x^*P^{-1}x, \quad \text{for all } x \in \mathbb{C}^n.$$

10. State the Rayleigh quotient iteration. Apply it to the 2×2 real matrix

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

where $\lambda_1 \neq \lambda_2$. Find the subset $S \subseteq \mathbb{R}^2$ having the property that the iteration applied to this matrix with initial guesses in S do not converge. Is S a set of measure zero?