Numerical Analysis
Preliminary Exam
May 17, 2001

Do any 8 of the following 10 problems.

Part 1: Numerical Linear Algebra

1. (a) Under what conditions does a real n by n matrix $\mathbf{A}$ have a Schur decomposition $\mathbf{A} = \mathbf{U}^T\mathbf{T}\mathbf{U}$, where $\mathbf{U}$ and $\mathbf{T}$ are real n by n matrices with $\mathbf{U}$ orthogonal and $\mathbf{T}$ upper triangular?
(b) Under what conditions on $\mathbf{A}$ is $\mathbf{T}$ diagonal?
(c) Does the matrix
$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$
have an orthogonal set of eigenvectors?
(d) Give a careful statement of the singular value decomposition of a real m by n matrix.
(e) Under what conditions does the singular value decomposition exist?

2. Consider the linear system $\mathbf{Ax} = \mathbf{b}$.
(a) If the rows of $\mathbf{A}$ are linearly independent, derive a formula for the solution of minimal 2-norm.
(b) If the columns of $\mathbf{A}$ are linear independent, derive a formula for the $\mathbf{x}$ that minimizes the 2-norm of the residual $\mathbf{r} = \mathbf{b} - \mathbf{Ax}$.
(c) Use the singular value decomposition of $\mathbf{A}$ to give a formula for the $\mathbf{x}$ that minimizes the 2-norm of the residual, and among all the $\mathbf{x}$'s which minimize the 2-norm, it has minimal norm.

3. (a) Given a vector $\mathbf{x} \in \Re^n$ and a natural number $k < n$, give a formula for a unit vector $\mathbf{w}$ for which the vector $\mathbf{y} = (\mathbf{I} - 2\mathbf{w}\mathbf{w}^T)\mathbf{x}$ satisfies the following conditions: $y_i = x_i$ for $i < k$ and $y_i = 0$ for $i > k$.
(b) Using these Householder transformations, write a pseudo code (or a matlab code) for reducing a real symmetric matrix to tridiagonal form using Householder similarity transformations.

4. (a) State and prove the Gerschgorin Circle Theorem.
(b) Let
$$\mathbf{A} = \begin{pmatrix} 5 & 1 & 3 \\ 2 & 4 & 1 \\ 3 & -1 & 4 \end{pmatrix}.$$
What can you say about the location of the eigenvalues of $\mathbf{A}$?
(c) Use the Gerschgorin Circle Theorem to estimate the size of the largest root of the polynomial $p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1x + a_0$.

5. (a) Find the ellipse or hyperbola of the form $ax^2 + by^2 = 1$ that best fits n data points $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$ in the plane by computing the least solution to the n linear equations gotten by substituting each of the data points into the quadratic equation.
(b) Test your method with the dataset $(1,1), (2,4), (3,9), (4,16)$.

Part II: Numerical Analysis

6. (a) State Newton's method (the algorithm) for solving $f(x) = 0$ where $f : R \to R$.
(b) Assuming f is smooth and we have an initial guess sufficiently close to a root p, state sufficient condition(s) for the method to converge quadratically.
(c) Show that Newton's method applied to $x^5 = 2$ converges quadratically to the root $2^{1/5}$ from any starting guess $x > 0$.

7. (a) Given a function f with $n+1$ continuous derivatives on the interval $I = [-1, 1]$, show that, given $n+1$ points $\{x_0, x_1, \ldots, x_n\}$ in I, there exists a unique polynomial $P_n(x)$ of degree $\leq n$ such that

$$P_n(x_i) = f(x_i) \quad \text{for} \quad i = 0 \ldots n.$$

(b) Prove that, given $t \in I$, there exists $\eta \in I$ such that

$$f(t) - P_n(t) = \frac{(t - x_0) \cdots (t - x_n)}{(n+1)!} f^{(n+1)}(\eta).$$

(c) From the above we get that (all norms are on the interval I)

$$\|f - P_n\|_\infty \leq \max_{t \in I} |(t - x_0)(t - x_1) \cdots (t - x_n)| \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}.$$

What choice for the points $x_0, \ldots, x_n$ minimizes the right hand side of this error bound?

8. (a) Find a function $q(x)$ which has a continuous first derivative for each $x \in (-\infty, \infty)$ with the properties:

$$q(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{for all } |x| \geq 2 \\ a_0 + a_1 x + a_2 x^2 & \text{for all } x \in [-2, -1] \\ b_0 + b_1 x + b_2 x^2 & \text{for all } x \in [-1, 1] \\ c_0 + c_1 x + c_2 x^2 & \text{for all } x \in [1, 2] \end{cases}$$

(b) Use the function $q(x)$ defined in part a to interpolate a set of equally spaced points $(x_0, y_0), (x_1, y_1), ..., (x_n, y_n)$, where $h = x_{k+1} - x_k$ for all k. In particular, describe how to find constants c_k so that the function $Q(x) = \sum_{k=0}^{n} c_k q(\frac{x - x_k}{h})$ has the property that $Q(x_k) = y_k$ for all k.

9. Let $P_2(x)$ be a quadratic polynomial interpolating $g(x)$ at $x = 0, h, 2h$.
(a) Use this to derive a numerical integration formula I_h for

$$I = \int_0^{3h} g(x)dx.$$

(b) Use a Taylor series expansion of $f(x)$ to show

$$I - I_h = 3/8h^4 f^{(3)}(0) + O(h^5).$$

10. **Triple Recursion Formula** If $\{\phi_n(x)\}$ is an orthogonal family of polynomials on $[a, b]$, with respect to the weight function $w(x) \geq 0$, and $n \geq 1$, then show that $\phi_{n+1}(x) = (a_n x + b_n)\phi_n(x) - c_n \phi_{n-1}(x)$, for some constants a_n, b_n, and c_n. (Hint: Let $g(x) = \phi_{n+1}(x) - a_n x \phi_n(x)$, where a_n is a constant chosen so that $g(x)$ has degree n.)