Numerical Analysis
Preliminary Exam
September 5, 2000

Do any 8 of the following 10 problems.

Part 1: Numerical Linear Algebra

1. A square matrix A which can be decomposed into a form $A = SDS^{-1}$, where D is diagonal is said to be diagonalizable.
(a) Is the matrix
$$A = \begin{pmatrix} 5 & 1 \\ 0 & 3 \end{pmatrix}$$
diagonalizable? Why?
(b) Can the above matrix A be diagonalized with a matrix S such that $S^*S = I$?

2. (a) State a procedure for factoring a matrix A (without pivoting) into a form $A = LDU$, where L is unit lower triangular, U is unit upper triangular, and D is diagonal.
(b) State sufficient conditions on A for the procedure in part (a) (without pivoting) to work.
(c) What is the computational advantage of having lower and upper triangular matrices? An explanation without specific computation counts is sufficient.

3. (a) Under what conditions can a matrix be factored into a form $A = U^*TU$, where $U^*U = I$, and T is upper triangular?
(b) If A is Hermitian, how is U constructed? (You don't need to actually give an algorithm)

4. (a) State the power method algorithm for finding a dominant eigenvalue and an associated eigenvector of a matrix A. (An eigenvalue λ_i is said to be dominant if $|\lambda_i| > |\lambda_k|$ for all other k.)
(b) Show that the power method converges (for almost any starting value) when A is real symmetric and has a dominant eigenvalue.
(c) Explain how the inverse power method can be used to find non-dominant eigenvalues and eigenvectors for a matrix A.

5. (a) Give a careful statement of the Gerschgorin Circle Theorem.
(b) Give a careful proof of the Gerschgorin Circle Theorem.
(c) Let
$$A = \begin{pmatrix} 5 & .01 & .03 \\ .02 & 4 & .01 \\ .03 & -.01 & 4 \end{pmatrix}.$$
What can you say about the location of the eigenvalues of A?

Part II: Numerical Analysis

6. (a) State Newton's method (the algorithm) for solving $f(x) = 0$ where $f : R \to R$.
(b) Assuming f is smooth and we have an initial guess sufficiently close to the root, state sufficient condition(s) for the method to converge quadratically.
(c) How would you use Newton's method to compute $5^{1/3}$?

7. (a) Give an exact expression for a Lagrange polynomial of degree n which would interpolate $n+1$ given data points, $f(x_k)$.
(b) Describe the differences between Hermite polynomials and Lagrange polynomials.
(c) Suppose we have a set of Lagrange polynomials of degree $2n+1$ which interpolate a smooth function at the points $x_0, x_0 + h, x_1, x_1 + h, x_2, x_2 + h, ..., x_n, x_n + h$. What is the pointwise limit of this interpolant as $h \to 0$?

8. (a) Describe Romberg's technique for numerical approximation of an integral (based on the trapezoid rule).
(b) Give a brief justification of the method.

9. **Triple Recursion Formula** If $\{\phi_n(x)\}$ is an orthogonal family of polynomials on $[a, b]$, with weight function $w(x) \geq 0$, and $n \geq 1$, then show that

$$\phi_{n+1}(x) = (a_n x + b_n)\phi_n(x) - c_n \phi_{n-1}(x),$$

for some constants a_n, b_n, and c_n. (Hint: Let $g(x) = \phi_{n+1}(x) - a_n x \phi_n(x)$, where a_n is a constant chosen so that $g(x)$ has degree n.)

10. Let $\{\phi_k\}$ be a system of orthogonal polynomials, with respect to a non-negative weight $w(x)$, on $[a, b]$. Show that each of the polynomials $\phi_k(x)$ must have exactly k simple zeros in $[a, b]$.