Ph.D. Numerical Analysis Exam January 2000

Do any 8 of the following 10 problems.

1.

(1) Show: If a matrix $A \in R^{n\times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $\lambda_i \neq \lambda_j$ for all $i \neq j$, then the corresponding eigenvectors are linearly independent.
(2) Show: If a symmetric matrix $A \in R^{n\times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $\lambda_i \neq \lambda_j$ if $i \neq j$, then the eigenvectors of A are orthogonal.
(3) Show: If the matrix $A \in R^{n\times n}$ has n independent eigenvectors, then the matrix can be diagonalized.
(4) Show: If $A \in R^{n\times n}$ is symmetric, then the then the matrix can be diagonalized by an orthogonal matrix.

2.

(1) Give a careful statement of the singular value decomposition for an arbitrary matrix $A \in R^{m\times n}$.
(2) Give a careful proof of the singular value decomposition.
(3) Show how the singular value decomposition for A naturally yields a solution to the least squares problem $\min_x ||Ax - y||_2$.

3.

(1) Show: If w is a unit vector in R^n, then the matrix $H = I - 2ww^t$ is an orthogonal projection.
(2) Show: If $w = \frac{x-y}{\|x-y\|}$, $\|x\| = \|y\|$, and $H = I - 2ww^t$, then $Hx = y$.
(3) Given an arbitrary matrix $A \in R^{m\times n}$, construct an orthogonal matrix $Q \in R^{m\times m}$ and an upper triangular matrix $R \in R^{m\times n}$ such that $A = QR$.

4.

(1) Show: If λ_n is the unique eigenvalue of $A \in R^{n\times n}$ with largest magnitude and x_0 is chosen arbitrarily in R^n, then the iteration $x_k = A^k x_0$, will almost surely converge to a multiple of v_n, the eigenvector associated with λ_n.
(2) If λ_k is a unique eigenvalue which does not have the largest magnitude from among the eigenvectors of A, then describe a method that will compute the eigenvector corresponding to λ_k.
(3) Describe the QR iteration algorithm for computing the eigenvalues for a matrix $A \in R^{n\times n}$. Draw analogues between the QR method and the power method.

5.

(1) If $A = D + F$, where $A, D, F \in R^{n\times n}$ and $D = diag(A)$, then state and prove a theorem relating the location of the eigenvalues of A to the matrices D and F.

(2) Show: If $A, D, F \in R^{n\times n}$ $Ax'(0) + Fx = \lambda'(0)x + \lambda x'(0)$.

(3) Using part (2) above show that $|\lambda'(0)| = \|\frac{y^*Fx}{y^*x}\| \le \frac{1}{y^*x}$.

(4) Explain the implications of (c) to the stability of eigenvalues of symmetric matrices, and highly unsymmetric matrices.

(BACKGROUND: It can be shown that for ϵ small there are differentiable functions $x(\epsilon)$, and $\lambda(\epsilon)$ such that $(A + \epsilon F)x(\epsilon) = \lambda(\epsilon)x(\epsilon)$, where $\|F\|_2 = 1$. Let $x = x(0)$ be a right eigenvector of A, and y be a left eigenvector of A.)

6.

(1) Give a careful statement of the error formula for Lagrange interpolation.

(2) Give a careful proof of the error formula for Lagrange interpolation.

(3) Give a careful definition of the Chebyshev Polynomials $T_n(x)$.

(4) Give a careful statement of the theorem that shows that the roots of $T_n(x)$ provide the optimal choice on $[-1, 1]$ for Lagrange interpolation.

(5) For the function $f(x) = \sin(\pi x)$ defined on $[-1, 1]$ and polynomial interpolating function $p_n(x)$ with interpolating points taken to be the roots of $T_n(x)$, and tolerance $\epsilon = 0.00001$, find an integer n such that the interpolating polynomial $p_n(x)$ has the property that $|p_n(x) - f(x)| < \epsilon$ for all $x \in [-1, 1]$.

7.

(1) Devise a 2nd order method for computing the cube root of a number.

(2) Show that your method always works.

8.

(1) Explain Romberg's technique for numerical approximation of an integral.

(2) Discuss the ideas and concepts that explain why the method works.

9.

(1) Set up the system of linear equations to be solved when a finite difference approach is used to solve the two point boundary value problem: $v'' + b(x)v' + c(x)v = d(x)$, where $v(0) = 0$ and $v(1) = 0$.

(2) Set up the system of linear equations to be solved when a collocation approach (with basis functions $\sin(k\pi x)$) is used to solve the two point boundary value problem: $v'' + b(x)v' + c(x)v = d(x)$, where $v(0) = 0$ and $v(1) = 0$.

10.

(1) Describe the defining properties of the Legendre polynomials on the interval $[-1, 1]$.

(2) Give a careful statement of the theorem fundamental to Gauss Quadrature.

(3) Show how the Gauss Quadrature method can be used to approximate the integral $\int_0^1 exp(x^2)dx$.