Ph.D. Numerical Analysis Exam — January 1996

1. Let A be a symmetric, positive definite matrix. Show that if A is LU factored, then the elements of U in absolute value are bounded by the largest diagonal element of A.

2. Suppose the columns of a matrix A are independent. Given the QR factorization of A, give a formula for the distance from some given vector b to the space spanned by the columns of A.

3. Give a careful statement and proof of Gershgorin's Theorem. Find the upper and lower bounds for the set of eigenvalues for the 4×4 Hilbert matrix having $(i, j)-$th entry $(i + j - 1)^{-1}$.

4. Explain how the singular values of a matrix can be used to determine the rank of a matrix in the presence of rounding errors.

5. Define the term "algorithm". Explain what is meant by saying that an algorithm is well conditioned and numerically stable.

Carefully formulate a *numerically stable* algorithm for computing the positive solution of the quadratic equation $x^2 + 2px - q = 0$ where p and q are arbitrary positive numbers. Justify the numerical stability of your algorithm.

6. Determine the cubic spline $s(x)$ on the grid $\{-1, 0, 1\}$ with $s(-1) = s(0) = 0$, $s(1) = 4$, and $s''(-1) = s''(1) = 0$.

7. For $h > 0$, let $T(h)$ be the trapezoidal approximation of $\tau = \int_a^b f(x)dx$, using an equi-spaced grid of step size h.

(a) What is the order of the error in this approximation as $h \to 0$?

(b) Assuming that $T(h)$ can be expressed as a power series of even powers of h, show how a more accurate approximation can be obtained by using $T(h)$ and $T(h/2)$. Express this approximation in terms of $T(h), T(h/2)$. (That is, you need to verify the first step in Romberg integration.)

(c) Describe the method of Romberg integration. Formulate it in terms of a table (Neville type).

8. Let A be a nonsingular $n \times n$ matrix and $\{X_k\}$ a sequence of $n \times n$ matrices generated by the iteration:

$$X_{k+1} = X_k + X_k(I - AX_k).$$

Show that if X_0 is chosen such that the spectral radius of $I - AX_0$ is strictly less than 1, then $\{X_k\}$ converges to the inverse of A.

9. State Euler's method for approximating the solution to the differential equation

$$\frac{dx}{dt} = f(x(t)), \quad x(0) = x_0.$$

Show that the error in Euler's method is $O(\Delta t)$, as $\Delta t \to 0$, for t sufficiently small, where Δt is the step size.

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