University of Florida

MAD6407 EXAM January 6, 2017

Name:

ID #:

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Directions: You have 2 hours to answer the following questions. You must show all your work as neatly and clearly as possible and indicate the final answer clearly. You may not use a calculator.

Problem	Possible	Points
1	25	
2	15	
3	20	
4	20	
5	20	
Total	100	

(1) (25 points)

(a) Show that the equation $x = \frac{1}{2}\cos(x)$ has a solution α.

(b) Find an interval $[a, b]$ containing α and such that for every $x_0 \in [a, b]$, the iterative sequence

$$x_{n+1} = \frac{1}{2}\cos x_n \tag{1}$$

will converge to α. Justify your answer.

(2) (15 points) For the basic Lagrange polynomials

$$L_i(x) = \Pi_{j \neq i} \frac{x - x_j}{x_i - x_j} \qquad \text{for} \qquad i = 0, \ldots, n,$$

show that

$$\sum_i L_i(x) = 1 \qquad \text{for} \quad \text{all} \qquad x.$$

(3) (20 points) Consider a quadrature rule of the form for $0 < \alpha < 1$:

$$\int_0^1 x^\alpha f(x)\, dx \approx A \int_0^1 f(x) dx + B \int_0^1 x f(x) dx.$$

(a) Determine the constants A, B so that the quadrature formula has maximum degree of exactness.

(b) What is the degree of exactness of the formula in part (a)?

(4) (20 points) Let f be an arbitrary (continuous) function on $[0, 1]$ satisfying

$$f(x) + f(1 - x) \equiv 1 \qquad for \qquad 0 \leq x \leq 1.$$

(a) Show that $\int_0^1 f(x)dx = \frac{1}{2}$.

(b) Show that the composite trapezoidal rule for computing $\int_0^1 f(x)dx$ is exact.

(5) (20 points) Assume that you are solving the initial value problem.

$$\begin{aligned} &y' = f(t, y) \qquad && a \le t \le b \\ &y(a) = \alpha && \end{aligned}$$

The formula for the global error of the numerical solutions for the ODE problem above obtained via Euler's method is ($M = ||Y''||_\infty$):

$$|Y(t_i) - w_i| < \frac{hM}{2L}[e^{L(b-a)} - 1].$$

Compute the values of L and $M = ||Y''||_\infty$ necessary to apply the global error formula above to the specific ODE problem

$$\begin{aligned} &y' = \sin(t + 2y) + e^t \qquad && 0 \le t \le 1 \\ &y(0) = 0 && \end{aligned}$$