Ph.D. Qualifying Examination

in

Differential Geometry

Summer 1992

Answer three questions from Section I and each question in Section II.

SECTION I

1. State the preimage (or regular value) theorem.
   Let M be the vector space of $m \times m$ real matrices and S its subspace of symmetric matrices. By considering the function
   $$F : M \longrightarrow S : A \longmapsto A^t A$$
   or otherwise, show that the orthogonal group $O(m)$ is naturally a submanifold of M having dimension $\frac{1}{2}m(m-1)$. With which subspace of M is $T_I O(m)$ naturally identified?

2. What are the integral curves of a vector field. What does it mean to say that a vector field is complete?
   Show that $\xi = y^2 \frac{\partial}{\partial x}$ and $\eta = x^2 \frac{\partial}{\partial y}$ are complete vector fields on $\mathbf{R}^2$ but that $\xi + \eta$ is not complete. Can the Lie bracket of two different incomplete vector fields be complete? Justify.

3. How is the exterior derivative $d\omega \in \Omega^{k+1}(M)$ of $\omega \in \Omega^k(M)$ defined?
   Let ω be a volume form on the orientable manifold M and let ξ be a vector field on M. Show that $L_\xi \omega$ is exact and that $L_\xi \omega = f_\xi \omega$ for some smooth function $f_\xi \in C(M)$ on M. Find an explicit formula for f_ξ if
   $$\xi = a_1 \frac{\partial}{\partial x_1} + \ldots + a_m \frac{\partial}{\partial x_m}$$
   on $M = \mathbf{R}^m$ with $a_1, \ldots, a_m \in C(\mathbf{R}^m)$ and $\omega = dx_1 \wedge \ldots \wedge dx_m$ in the usual coordinates. Hence suggest a suitable name for f_ξ in general.

4. Let $\omega = x dy - y dx$ be the standard one-form on the unit circle S^1. Let $f : \mathbf{R} \longrightarrow S^1$ be the inverse of stereographic projection from (0, 1). Compute $f^*\omega$ and hence show that
   $$\int_{S^1} \omega = 2\pi.$$
   How does this calculation establish that ω is not exact on S^1? Is $f^*\omega$ exact on $\mathbf{R}$? Why?

SECTION II

A. Explain what is meant by a Lie group G, its Lie algebra $\mathfrak{g}$, and the adjoint representation Ad of G on $\mathfrak{g}$. How does the adjoint representation of $Gl(m, \mathbf{R})$ appear in terms of the natural identifications? Show that if G is connected then the kernel of the adjoint representation coincides with the centre of G.
[It may be assumed that if $t \in \mathbf{R}$, $g \in G$ and $\xi \in \mathfrak{g}$ then

$$\exp(tAd_g\xi) = g(expt\xi)g^{-1}.]$$

B. Let (M, ω) be a symplectic manifold. How is the Hamiltonian vector field ξ_H of $H \in C(M)$ defined? How is the Poisson bracket defined on $C(M)$? Show that if $(p_1, ..., p_m, q_1, ..., q_m)$ are (local) symplectic coordinates on M then the integral curves of ξ_H satisfy the Hamilton equations

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}.$$

Show also that the Poisson centralizer of $H \in C(M)$ is precisely the set of all functions on M constant along integral curves of ξ_H and suggest a mechanical interpretation of this result.

C. Define a geodesic in terms of the Levi-Civita connexion ∇ on the Riemannian manifold (M, g) and give the form taken by the geodesic equation in local coordinates. For the metric g on the open upper half-plane M having $\{y\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\}$ as orthonormal vector fields, show that the nonzero Christoffel symbols are

$$\Gamma^y_{yy} = \Gamma^x_{xy} = \Gamma^x_{yx} = -\Gamma^y_{xx} = -\frac{1}{y}.$$

Hence show that vertical upper half-lines and upper semicircles centered on the x-axis are geodesics.
[It may be assumed that if ξ, η, ζ are vector fields on M then

$$2g(\nabla_\xi \eta, \zeta) = \xi \cdot g(\eta, \xi) + \eta \cdot g(\xi, \zeta) - \zeta \cdot g(\xi, \eta) \\ - g(\xi, [\eta, \zeta]) - g(\eta, [\xi . \zeta]) + g(\zeta, [\xi, \eta]).]$$