Numerical Analysis Exam: August, 2019
Do **4** (four) problems.

1. Consider using Newton's method to find $x \in \mathbb{R}^n$ such that $F(x) = 0$, for $F : D \subseteq R^n \to \mathbb{R}^n$, where D is open and convex and F is continuously differentiable on D.

Suppose x^* satisfies $F(x^*) = 0$; and $J(x)$, the Jacobian of F evaluated at x satisfies $\|J^{-1}(x)\| \leq \mu$ for some number $\mu > 0$ for all x in a convex neighborhood $N \subseteq D$ that contains x^*.

(a) Assuming there is a constant $0 < \theta < 1$ for which $\|J(x) - J(y)\| \leq \theta/\mu$ for all $x, y \in N$, show that $\{x_k\}$ converges at least linearly to x^* whenever $x_0 \in N$.

(b) Assuming there is a constant κ such that $\|J(y) - J(x)\| \leq \kappa\|y - x\|$ for each $x, y \in N$, show that $\{x_k\}$ converges quadratically to x^* whenever $x_0 \in N$.

2. Consider the data points $(x_1, y_1) = (0, -1)$, $(x_2, y_2) = (1, 3)$, $(x_3, y_3) = (2, 2)$.

(a) Construct the both the Newton and Lagrange forms of the interpolating polynomial through (x_1, y_1), (x_2, y_2), (x_3, y_3) (each can be left in the form of a linear combination of basis functions, but each basis function and coefficient should be explicitly shown).

(b) Let $f(x) = 1/x$ and show for $x_0, \ldots, x_n \neq 0$ that

$$f[x_0, x_1, \ldots, x_n] = (-1)^n \prod_{i=0}^{n} \frac{1}{x_i}.$$

3. The third Chebyshev polynomial T_3 is given by $T_3(x) = 4x^3 - 3x$.

(a) Find the Chebyshev points on $-2 \leq x \leq 2$.

(b) Find the gobal interpolating polynomial on $-2 \leq x \leq 2$ that interpolates $f(x) = e^x$ at the Chebyshev nodes.

(c) Suppose you were given 200 pieces of data, equally spaced on $-2 \leq x \leq 2$, and you want to approximate values that lie between data points. Explain what kind of interpolant you would use to fit these data, and why.

4. Consider the function

$$s(x) = \begin{cases} s_1(x) = (x+1)^3 - 3x, & -1 \leq x \leq 0 \\ s_2(x) = (1-x)^3 + 3x, & 0 \leq x \leq 1 \end{cases}$$

(a) Is $s(x)$ a cubic spline? If so, what kind?

(b) The trapezoid rule for numerical integration over $a \leq x \leq b$ has an error given by

$$\int_a^b f(x)\,dx = I_T - \frac{1}{12}(b-a)^3 f''(\eta), \qquad \text{for some } \eta \in (a, b).$$

Determine a bound for the error in the composite trapezoid rule assuming $[a, b]$ is broken up into n equally spaced intervals of length $h = (b - a)/n$.

(c) Use either the midpoint rule or the trapezoid rule with $n = 1$ then $n = 2$ to approximate $\int_{-1}^{1} s(x)\,dx$. Explain which rule you chose to use and why.

5. Let $x_0 = a$, $x_1 = a + h$ and $x_2 = b = a + 2h$, and let $f \in C^2[a, b]$.

(a) Construct the difference approximation to $f''(x_1)$ based on the $\mathcal{P}_2$ interpolant of f on $[a, b]$ with interpolation points x_0, x_1, x_2 (you should explicitly show how the difference approximation is derived from the interpolant).

(b) The p_2 difference approximation satisfies $|f''(x_1) - p_2''(x_1)| \leq \frac{Mh^2}{12}$, where M is a constant that depends on f. Suppose the data is noisy and the approximation is based on the values f_i where $f_i - f(x_i) = \epsilon_i$ with $|\epsilon_i| < \epsilon, \quad i = 0, 1, 2$, for a given value of ϵ. What is the best accuracy with which $f''(x_1)$ can be approximated? For what value of h is it attained?