Numerical Linear Algebra Exam: May, 2019

Do **4** (four) problems.

1. **(a)** Show the matrix norm equality for $A \in \mathbb{C}^{m\times n}$

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

(b) Explain why the matrix 1-norm, 2-norm and ∞-norm are the most commonly used of the matrix p-norms in scientific computing.

(c) Show $\rho(A) \le \|A\|$ where $\|A\|$ is any subordinate (induced) matrix norm and $\rho(A)$ is the spectral radius of A.

2. Let $A \in \mathbb{C}^{m\times n}$ with $\text{rank}(A) = n < m$. Let $A = QR$ be the QR decomposition of A, and $A = Q_1R_1$ be the economy QR decomposition.

(a) Show $Q_1Q_1^*$ is an orthogonal projector onto $\text{Col}\,(A)$.

(b) Let $b \in \mathbb{C}^m$. Write down an expression for the least-squares solution to $Ax = b$ as the solution to an $n \times n$ system in terms of Q_1, (and/or Q_1^*), R_1, x and b.

3. Let $A = U\Sigma V^*$ be the singular value decomposition of $A \in \mathbb{C}^{m\times n}$ with $\text{rank}\,(A) = p$ and $p \le n \le m$.

(a) Show $\text{Col}\,(A^*) = \text{Span}\,\{v_1, v_2, \ldots, v_p\}$, where $v_1, \ldots, v_p$ are the first p columns of V.

(b) Show $\text{Null}\,(A) = \text{Span}\,\{v_{p+1}, v_{p+2}, \ldots, v_n\}$.

(c) Suppose the right singular vectors $v_1, \ldots, v_p$ have been computed. Describe how to compute the left singular vectors $u_1, \ldots, u_p$ (without solving a spectral problem).

4. Let $\|\cdot\|$ be a subordinate (induced) matrix norm. If A is $n \times n$ invertible and E is $n \times n$ with $\|A^{-1}\|\|E\| < 1$, then show

(a) $A + E$ is nonsinguar

(b)

$$\|(A+E)^{-1}\| \le \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|E\|}.$$

5. Consider the matrix A given by

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ -1 & 4 & -1 & 1 \\ 2 & -1 & 6 & -2 \\ 0 & 1 & -2 & 4 \end{pmatrix}$$

Suppose the eigenvalues of A are all distinct (they are) and satisfy $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$.

(a) Show that A is positive definite.

(b) Describe an algorithm that could be used to converge to λ_4.

(c) Describe an algorithm that could be used to converge to λ_2.

Numerical Analysis Exam: May, 2019

Do **4** (four) problems.

1. Consider the fixed point problem $x = f(x)$ where $f(x) = e^{-(2+x)}$.

(a) Find the largest open interval in $\mathbb{R}$ where $f(x)$ is a contraction.

(b) Assuming all computations are done in exact arithmetic, find the largest open interval in $\mathbb{R}$ where the fixed-point iteration $x_{k+1} = f(x)$ is assured to converge.

(c) Write a Newton iteration for finding the fixed-point.

2. Let $x_1, x_2, \ldots, x_{n+1}$ be $n+1$ distinct numbers. Let $l_j(x)$ be the associated Lagrange basis polynomials, $j = 1, \ldots n+1$.

(a) State the definition of $l_j(x)$ and show that $\{l_j(x)\}_{j=1}^{n+1}$ form a basis for $\mathcal{P}_n$, the space of polynomials of degree at most n.

(b) Show that

$$\sum_{j=1}^{n+1} (x - x_j)^k l_j(x) = 0, \qquad \text{for all } \; k = 1, \ldots, n.$$

3. Consider the interval $[a, b]$ with the partition $a = x_1 < x_2 < \cdots < x_n < x_{n+1} = b$. Suppose $s(x)$ is the natural cubic spline that interpolates the data $\{(x_i, y_i)\}_{i=1}^{n+1}$, and that $g \in C^2[a, b]$ interpolates the same data. Show that

$$\int_a^b (s''(x))^2 \, \mathrm{d}\, x \leq \int_a^b (g''(x))^2 \, \mathrm{d}\, x.$$

4. **(a)** Consider the inner product on $C(0, 2)$ given by $(f, g) = \int_0^2 f(t)g(t) \, \mathrm{d}\, t$. Find three orthonormal polynomials ϕ_0, ϕ_1, ϕ_2 on $(0, 2)$ with respect to the given inner product such that the degree of ϕ_n is equal to n, $n = 0, 1, 2$.

(b) Find the nodes t_1 and t_2 and weights w_1 and w_2 which yield the weighted Gaussian Quadrature formula

$$\int_0^2 f(t) \, \mathrm{d}\, t \approx w_1 f(t_1) + w_2 f(t_2)$$

with degree of exactness $m = 3$. **You should find the nodes exactly, and may leave the weights w_1, w_2 in integral form.**

5. Let $f \in C^\infty(a - H, a + H)$, and let $h < H$. Let $x_0 = a - h$, $x_1 = a$ and $x_2 = a + h$.

(a) Find the finite difference approximation to $f'(a)$ based the interpolant p_2 which satisfies $p_2(x_0) = f(x_0)$, $p_2(x_1) = f(x_1)$ and $p_2(x_2) = f(x_2)$.

(b) Let $\psi_0(h) = \psi(h)$ be the difference approximation to $f'(a)$ found in part **(a)**. Assume (in exact arithmetic) $\psi(h) \to \psi(0) = f'(a)$ as $h \to 0$, and that $\psi(h)$ has the asympototic expansion

$$\psi(h) = \psi(0) + a_2 h^2 + a_4 h^4 + a_6 h^6 + \ldots.$$

Find the general Richardson extrapolation formula for $\psi_k(h)$ based on $\psi_{k-1}(h)$ and $\psi_{k-1}(h/2)$.