Numerical Analysis Qualifying Exam – May, 2017

Do all five (5) problems

1. With

$$\phi(w,t) = af(w+ch, t+bh),$$

find the values of the parameters a, b, c so that the resulting one-step method

$$w_0 = \alpha$$
$$w_{i+1} = w_i + h\phi(w_i, t_i)$$

has local truncation error $O(h^2)$

2. Hint:the error term in the *one unit* Trapezoid rule is $-\frac{h^3}{12}f''(\eta)$ and in the *one unit* Simpsons' rule is $-\frac{h^5}{90}f^{(4)}(\eta)$.

Let S be the cubic spline given by

$$S(x) = (x+1)^3 \text{ for } x \in [-1, 0]$$
$$S(x) = (1-x)^3 \text{ for } x \in [0, 1]$$

(a) Estimate the error of the composite trapezoidal rule applied to $\int_{-1}^{1} S(x)\ dx$, when $[-1, 1]$ is divided into n subintervals of equal length $h = 2/n$ and n is even (and so 0 is a node).

(b) Estimate the error of the composite Simpson's rule applied to $\int_{-1}^{1} S(x)\ dx$, when $[-1, 1]$ is divided into n subintervals of equal length $h = 2/n$ and n is divisible by 4 (and so 0 is a node).

3. (a) If $f \in C^1[a,b]$ and $a \leq x_0 < \cdots < x_n \leq b$ and H, G are degree at most $2n+1$ polynomials with $G(x_i) = H(x_i) = f(x_i)$ and $G'(x_i) = H'(x_i) = f'(x_i)$ for all i, then $G = H$.

(b) If $\varphi_0, \varphi_1, \ldots, \varphi_n$ are polynomials with φ_n of degree n, then the set of φ_i is linearly independent.

4. Consider the inner product on $C[0, \infty)$ given by

$$\langle f, g \rangle = \int_0^\infty f(x)g(x)e^{-x}\, dx$$

(a) Starting with the basis $\{1, t, t^2\}$ for $\mathcal{P}_2[0, \infty)$ find three orthonormal polynomials ϕ_0, ϕ_1, ϕ_2 on $[0, \infty)$ with respect to the inner product and the degree of ϕ_n is equal to n. Hint: $\int_0^\infty t^m e^{-t}\, dt = m!$.

(b) Find the equations satisfied by the values of w_1, w_2, t_1 and t_2 which yield the weighted Gaussian Quadrature formula

$$\int_0^\infty f(t)e^{-t}\, dt = w_1 f(t_1) + w_2 f(t_2)$$

with degree of precision 3.

5. (a) Assume $g \in C^2[a, b]$ with $g([a, b]) \subset [a, b]$ and fixed point $p \in (a, b)$. Assume that $g'(p) = 0$. Show that for any $x \in [a, b]$ with $x \neq p$

$$\frac{|g(x) - p|}{|x - p|^2} \leq M$$

where $M = \max\{|g''(z)| \colon z \in [a, b]\}/2$.

(b) Let $g(x) = x - \tan(x)$. Find a fixed point p of g with $g'(p) = 0$ and give an explicit $[a, b]$ where you prove that for all $x \in [a, b]$, $g^n(x) \to p$.

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