Second Semester Algebra Exam **May 11, 2018**

Answer four problems. You should indicate which problems you wish to have graded. Write your solutions clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let D be a positive integer such that $-D \equiv 1 \pmod 4$. Define

$$\omega = \frac{1+\sqrt{-D}}{2}, \qquad R = \{a + b\omega : a, b \in \mathbb{Z}\}.$$

   For $\alpha \in R$ define $\mathrm{N}(\alpha) = \alpha\overline{\alpha}$, where $\overline{\alpha}$ is the complex conjugate of α.

   (a) Prove that R is a subring of the field $\mathbb{C}$ of complex numbers.

   (b) Prove that for all $\alpha \in R$ we have $\mathrm{N}(\alpha) \in \mathbb{Z}_{\geq 0}$.

   (c) Prove that for all $\alpha, \beta \in R$ we have $\mathrm{N}(\alpha\beta) = \mathrm{N}(\alpha)\mathrm{N}(\beta)$.

   (d) Let $\alpha \in R$. Prove that α is a unit if and only if $\mathrm{N}(\alpha) = 1$.

2. (a) Let F be a field and let G be a finite subgroup of the unit group $F^\times$. Prove that G is cyclic.

   (b) Give an example of a field F and a finitely generated subgroup G of $F^\times$ which is not cyclic.

3. Let R be a commutative ring with $1 \neq 0$ and let M be a unital R-module. Say M is irreducible if $M \neq \{0\}$ and the only submodules of M are $\{0\}$ and M. Prove that if M is an irreducible R-module then $M \cong R/I$ for some maximal ideal I in R.

4. Let V be a vector space over a field F and let S be a subset of V such that $\mathrm{Span}(S) = V$. Use Zorn's lemma to prove that there is a subset $\mathcal{B}$ of S which is a basis for V.

5. Find a representative for each similarity class of 3×3 matrices A with entries in $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$ such that $A^4 = A$. (Note that $X^4 - X$ factors into linear factors over $\mathbb{F}_3$.)

6. Let $\gamma \in \mathbb{C}$ be a root of $X^3 - 6X^2 + 2 \in \mathbb{Q}[X]$. Express $(1+\gamma^2)(3-4\gamma^2)$ and $1/(1+\gamma^2)$ in the form $a + b\gamma + c\gamma^2$, with $a, b, c \in \mathbb{Q}$.