Numerical Linear Algebra Qualifying Exam – May, 2018
Do **4** (four) problems

1. (a) If P is a projector, prove that $\text{null}(P) \cap \text{range}(P) = \emptyset$ and $\text{null}(P) = \text{range}(I - P)$.
   (b) Prove that P is an orthogonal projector if and only if it is Hermitian.
   (c) If $q_1, \ldots q_n$ is an orthonornal basis for the subspace $V \subset \mathbb{C}^m$ with $m > n$, prove that the orthognal projector onto V is QQ^*, where Q is the matrix whose columns are the q_j.

2. Assume $A \in \mathbb{C}^{m \times m}$.
   (a) Prove that $\|A\|_2 = (\rho(A^*A))^{1/2} = \sigma_1$, where σ_1 is the largest singular value of A and ρ is the spectral radius.
   (b) Let $\kappa_2(A)$ be the two-norm condition number of the square, non-singular A. Prove that
   $$\kappa_2(A) = \frac{\sigma_1}{\sigma_m}.$$
   where σ_1 and σ_m are the largest and smallest singular values of A, respectively.
   (c) Show that $\kappa_2(A) = 1$ if and only if $A = rQ$ with $r \in \mathbb{R}$ and Q unitary.

3. (a) Given $A \in \mathbb{C}^{m \times n}$ with $m \geq n$, show that A^*A is nonsingular if and only if A has full rank.
   (b) If $u, v \in \mathbb{C}^m$ and $A = uv^*$, show that $\|A\|_2 = \|u\|_2 \|v\|_2$.

4. (a) Prove that every square matrix A has a Schur factorization.
   (b) If A is normal (so $A^*A = AA^*$) show that the triangular matrix in its Schur factorization is diagonal.

5. Assume $A \in \mathbb{R}^{m,m}$
   (a) Prove that $\langle x, y \rangle_A = x^*Ay$ is an inner product on $\mathbb{R}^m$ if and only if A is symmetric and positive definite
   (b) Assume now that A is symmetric and positive definite. If x_* is the solution to $Ax = b$ and $\{p_1, \ldots, p_m\}$ is an orthonormal basis for $\mathbb{R}^m$ with respect to $\langle\ ,\ \rangle_A$ and $x_* = \sum c_i p_i$, give a formula for the c_i.