Numerical Linear Algebra Qualifying Exam (January 2, 2014). *Answer any 4 questions.*

1. Suppose that $A \in \mathbb{C}^{n\times n}$ is a normal matrix; that is, $AA^* = A^*A$ where A^* is the conjugate transpose of A. Show that $A = X\Lambda X^*$ where Λ is diagonal and X is unitary; that is, show that A can be diagonalized and its eigenvector matrix is unitary.

2. (a) Suppose p and $q \in \mathbb{R}$ with p and q positive and $p^{-1} + q^{-1} = 1$. Show that for any matrix $A \in \mathbb{C}^{m\times n}$, we have $\|A\|_p = \|A^*\|_q$. Here $\|A\|_p$ denotes the matrix p-norm induced by the vector p-norm.

   (b) Prove that
   $$\|A\|_2^2 \leq \|A\|_p \|A\|_q$$
   for any $A \in \mathbb{C}^{m\times n}$ and any positive p and $q \in \mathbb{R}$ with $p^{-1} + q^{-1} = 1$.

   (c) Prove that for any $p \geq 1$ and any diagonal matrix $D \in \mathbb{C}^{n\times n}$, we have
   $$\|D\|_p = \max\{|d_{ii}| : 1 \leq i \leq n\}.$$

   (d) Show that $\|A\|_2$ is the largest singular value of A.

3. (a) For any matrices $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times n}$, show that the nonzero eigenvalues of AB and BA are the same.

   (b) If AB is normal, $\|\cdot\|_2$ is the 2-norm of a matrix, and $\|\cdot\|$ is an induced matrix norm, then show that $\|AB\|_2 \leq \|BA\|$.

4. (a) Show that the eigenvalues of a projector are either 0 or 1.

   (b) Show that a projector P is orthogonal if and only if $P = P^*$.

5. Suppose that $A \in \mathbb{C}^{n\times n}$ and consider the series
   $$S = \sum_{i=0}^{\infty} A^i.$$
   Show that the series is convergent if and only if the spectral radius of A is strictly smaller than 1. If the series is convergent, then show that S is invertible and $S = (I - A)^{-1}$.