Second Semester Algebra Exam **January 5, 2016**

Answer four problems, and on the list below circle the problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

Grade: 1 2 3 4 5 6 7

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1. Let R be a nonzero commutative ring with identity. Show that R has a minimal prime ideal, i.e. a prime ideal not containing any other prime ideal.

2. Which of the following polynomials are irreducible? Explain.

   (a) $X^3 + X + 1$ in $\mathbb{F}_2[X]$;

   (b) $X^3 + X + 1$ in $\mathbb{Q}[X]$;

   (c) $X^5 + 6X^2 + 9X + 3$ in $\mathbb{Q}[X]$;

3. Show that a nonzero ring R with identity (not necessarily commutative) is a division ring if and only if its only left ideals are 0 and R.

4. Let R be an integral domain. (i) Define what it means for an element of R to be *irreducible*. (ii) Define what it means for an element of R to be *prime*. (iii) Show that a prime element is irreducible.

5. Suppose F is a field and V is an F-vector space. Denote by V' the dual of V, i.e. the F-vector space of linear maps $V \to F$. (i) Show that if $\dim_F V < \infty$ then $V \simeq V'$. (ii) Show that if V has infinite dimension then $V \not\simeq V'$.

6. Show that if L/K and E/L are algebraic extendsions of fields, then so is E/K.

7. Show that $GL_3(\mathbb{F}_2)$ has two conjugacy classes of elements of order 7, and find representatives of each.