Second-Semester Algebra Exam **May 10, 2013**

Answer four problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let R be an integral domain, let $P \subset R$ be a prime ideal, and let $F = \mathrm{Frac}(R)$ be the field of fractions of R. Define
$$R_P = \left\{ \frac{a}{b} : a, b \in R,\ b \notin P \right\}.$$
   (a) Prove that R_P is a subring of F.
   (b) Prove that R_P has a unique maximal ideal.

2. Let $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{-3} \in \mathbb{C}$.
   (a) Prove that ω is a root of the polynomial $f(X) = X^2 + X + 1$.
   (b) Prove that the set $\mathbb{Z}[\omega] = \{a + b\omega : a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{C}$.
   (c) Prove that $\mathbb{Z}[\omega]$ is a Euclidean domain with respect to the norm $\mathrm{N}(\alpha) = \alpha\overline{\alpha}$, where $\overline{\alpha}$ denotes the complex conjugate of $\alpha \in \mathbb{Z}[\omega]$.

3. Let R be an integral domain and let $P \subset R$ be a prime ideal. Let
$$f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0.$$
be a monic polynomial of degree $n \geq 1$ with coefficients in R.
   (a) Prove Eisenstein's criterion: If $a_i \in P$ for $0 \leq i \leq n-1$ and $a_0 \notin P^2$ then $f(X)$ is irreducible in $R[X]$.
   (b) Give an example of $f(X)$ as above such that $a_i \in P$ for $0 \leq i \leq n-1$ but $f(X)$ is not irreducible.

4. (a) Let R be an integral domain and let M be an R-module. Define the rank of M.
   (b) Prove that $\mathbb{Q}$ has rank 1 as a $\mathbb{Z}$-module.
   (c) Prove that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

5. Find a representative for each similarity class of **noninvertible** 3×3 matrices with entries in the field $\mathbb{F}_2$ with 2 elements.

6. Let E/F be a field extension and let R be a subring of E which contains F.

   (a) Prove that if E/F is an algebraic extension then R is a field.

   (b) Give an example where E/F is not an algebraic extension and R is not a field.