Numerical Linear Algebra Exam – August, 2017
Do **4** (four) problems

1. Assume that A is Hermitian and all its eigenvalues are distinct and nonzero.
   (a) Show that each pair of distinct eigenvectors of A are othogonal.
   (b) If λ is an eigenvalue of A with eigenvector x with $\|x\|_2 = 1$, then $B = A - \lambda xx^*$ has the same eigenvectors as A while $B's$ eigenvalues are the same as those of A except λ is replaced with zero.
2. (a) If P is a projector, prove that $\text{null}(P) \cap \text{range}(P) = \emptyset$ and $\text{null}(P) = \text{range}(I - P)$.
   (b) Prove that P is an orthogonal projector if and only if it is Hermitian.
3. (a) If both A and U are in $\mathbb{C}^{m,m}$ and U is unitary, prove that $\|UA\|_F = \|AU\|_F = \|A\|_F$.
   (b) If both A and U are in $\mathbb{C}^{m,m}$ and U is unitary, prove that $\|UA\|_2 = \|AU\|_2 = \|A\|_2$.
   (c) Prove that $\|A\|_2 = (\rho(A^*A))^{1/2} = \sigma_1$, where σ_1 is the largest singular values of A.
4. (a) If $A \in \mathbb{C}^{m,n}$ with $m \geq n$, prove that A^*A is invertible if and only if $\text{rank}(A) = n$.
   (b) Give an explicit formula for $\det(\lambda I - ww^*)$ when $\lambda \in \mathbb{C}$, I is the $m \times m$ identity matrix and $w \in \mathbb{C}^m$
5. Assume that T is tridiagonal and symmetric wiith the diagonal entries given by a_j for $j = 1, \ldots, m$ and the super- and sub-diagonal entries by b_j for $j = 1 \ldots m - 1$. Let p_k be the characteristic polynomial of the $k \times k$ matrix in the upper left hand corner of A. Prove that
$$p_k(x) = (a_k - x)p_{k-1}(x) - b_{k-1}^2 p_{k-2}(x).$$