PH.D. QUALIFYING EXAM IN COMPLEX ANALYSIS

Give complete proofs and computations. Partial credit will be given where justified.

1) Evaluate the integral
\[ \int_{0}^{\infty} \frac{\ln x}{(1 + x^2)^2} \, dx. \]

2) Find the order of each of the following entire functions:
   (a) \( \frac{\sin \sqrt{z}}{\sqrt{z}} \)
   (b) \( \prod_{k=1}^{\infty} \left(1 + \frac{z^k}{k!}\right) \)
   (c) \( \prod_{k=1}^{\infty} \left(1 + \frac{z^k}{k \ln^2 k}\right) \)

3) Let \( G \subseteq \mathbb{C} \) be a region and \( \{f_n\} \subseteq H(G) \) be a sequence of injective functions which converges to \( f \) in \( H(G) \). Prove that either \( f \) is also injective or it is constant on \( G \).

4) Let \( f \) be an analytic function mapping the unit disc \( D \) into itself and having two or more distinct fixed points in \( D \). Show that \( f \) must be the identity function \( f(z) = z \) for all \( z \in D \).

5) Construct the analytic function which map the unit disc \( D \) conformally onto the angular region \( |\arg(z)| < a \), for fixed \( a \in (0, \pi) \), and satisfies \( f(0) = 1 \). With the help of this function, prove that if \( g \) is a function which is analytic in \( D \) and satisfies both \( g(0) = 1 \) and \( |\arg(g(z))| < a \) \( (a \in (0, \pi)) \), then \( |g'(0)| \leq 4a/\pi \).

6) Let \( f \) be an entire function of finite order. Prove that if the order is not an integer, then \( f \) must have infinitely many zeros. Does there exist an entire function of infinite order with finitely many zeros? Explain.

7) Let \( n \) be a natural number and \( a \) be a real number such that \( a > e \). Show that the equation \( e^x - ax^n = 0 \) has exactly \( n \) solutions inside the unit disc.

8) Suppose \( f \) is entire and that there exists a bounded sequence \( \{a_k\}_{k \in \mathbb{N}} \) of distinct real numbers such that \( f(a_k) \) is real for every \( k \). Prove that \( f(x) \) is real for all real \( x \).